DEPARTMENT OF APPLIED MATHEMATICS,
UNIVERSITY COLLEGE, UNIVERSITY OF LONDON

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TECHNICAL SERIES. VI.

ON A PRACTICAL THEORY OF ELLIPTIC AND PSEUDO-ELLIPTIC ARCHES, WITH SPECIAL REFERENCE TO THE IDEAL MASONRY ARCH

BY
KARL PEARSON, F.R.S., W. D. REYNOLDS, B.Sc. (Eng.)
AND
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WITH SIX PLATES REPRODUCED FROM ACTUAL DRAWINGS OF ARCHES AND TWO DIAGRAMS IN THE TEXT

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On a Practical Theory of Elliptic and Pseudo-Elliptic Arches, with special reference to the Ideal Masonry Arch.

(1) *Introduction.* The current treatment of the masonry arch cannot from any point of view be considered a satisfactory one. It is assumed that, owing to the complexity of the conditions at the abutments, the true line of pressure cannot be drawn. This true or actual line of pressure is then replaced for practical purposes by the so-called "critical line of pressure." If the arch be "stable" for the critical line of pressure, then it is asserted to be stable under the actual condition of affairs. The reasoning adopted being that the actual must pass through the critical line of pressure before the arch could collapse, and if at the stage of passing through the critical line there is stability, then it is assumed that there will not be further progress towards collapse. The "stability" of the line of pressure is determined by three tests: (i) it must lie entirely within the core, usually the middle third at each cross-section; (ii) it must make at each cross-section an angle less than the angle of friction with the normal to the cross-section; (iii) the maximum compressive stress must be less than the safe compressive strength of the material.

The actual construction of the critical line of pressure has been subject to a good deal of modification. The earlier writers took it as a link polygon for the system of loads, passing for symmetrical loading through the outer middle third at the crown and the inner middle third at the springings. Other authorities while preserving the outer middle third at the crown have adopted arbitrary points on the inner middle third towards the springings according as to whether the arch was circular or elliptic. Undoubtedly a much better rule is to make the critical line of pressure *touch* the extrados middle third at the crown and the intrados middle third towards the springings. This rule can be extended so as to apply to asymmetrical loading, although the constructions, as will be illustrated in another memoir of this series, become more elaborate. When the arch, instead of being flattish, becomes Gothic, we have to transform the rule and make the critical line of pressure pass through the apex on the intrados middle third at the crown and touch the extrados middle third towards the haunches.

Experimenting on arches built up of wooden blocks, we do find that these arches open up roughly at the sections suggested by the critical lines of pressure. Still the whole arch theory cannot be admitted to be in a satisfactory condition. The results reached are rarely fulfilled in practical design; one or at any rate two locomotives on almost any brick railway arch in this country, if asymmetrically placed will give an

unstable line of pressure, and the masonry by this criterion is in tension, and should be opening up at the joints. Further, if an arch with symmetrical loading be found theoretically unstable, the modification *ab initio* of the drawings is a troublesome piece of work. We really want to know before designing what is the right form to give an arch so that it shall have no bending moment, and although we know that it must be a link polygon for the system of loading, that loading in masonry work depends largely on the form of the arch and accordingly can only be determined *a posteriori*.

So unsatisfactory has been the theory of masonry arches that some continental writers have simply applied to them the theory of built-in metal arches. The abutment conditions and the variations in the elastic constants are so very different in the two cases that exception may well be taken to this practise, although it has led to some fine arches which satisfy with extraordinary economy the critical line of pressure conditions.

The object of the present memoir is to determine the dimensions of arches of a customary form, nearly elliptic, which will under the usual load systems *practically satisfy* the condition of no bending moment, and then to compare this test with the well-known critical line of pressure results. The elliptic brick arch has been largely used on some English railways on the ground that "it saves brickwork in the haunches" and so decreases the cost. If we raised the point that according to the critical line of pressure theory many such arches are unstable, the reply has been that the designer includes the backing as part of his arch proper. In such a case the masonry of the backing ought to be equal in character to that of the arch, and it may be doubted where then the economy would come in. It appears therefore desirable to work out by elementary means the general problem of the elliptic arch, asking for what systems of loading the centre lines of elliptic or allied arches practically coincide with their lines of pressure.

(2) *Ideal Arches.* The ideal arch is we know the arch of which the centre line is the actual line of pressure for its load.

Several cases of this are familiar to everybody:

(α) The parabola is the ideal arch for a uniform platform load.

(β) The catenary arch is the ideal arch for a uniform load per foot run of the rib itself.

For example, the ribs of a station roof designed to carry a practically uniform weight of roofing should be catenary arches.

(γ) The catenary arch is also the ideal arch to carry a uniform vertical load rising to its directrix.

For example, if an arch is to be pierced through a vertical wall of uniform height, the proper arch is a catenary with its directrix at the top of the wall.

It will be at once seen that (γ) is very limited for it fixes the load to the directrix of the catenary. If, to take the more general case, we superimpose a uniform platform load above the directrix, we clearly modify the catenary in the direction of the parabola.

Now it will be obvious that for any actual masonry arch we have not only

(i) a uniform platform load,

(ii) a uniform load per foot run of the rib,

(iii) a uniform vertical load rising to a horizontal line,

but also,

(iv) a varying vertical load rising from the extrados to the backing.

The shape of the backing may be of various types: it is sometimes bounded by two tangents to the extrados, having points of contact a little distance from the crown, there being practically no backing at the crown. In other cases it is determined by a circular arc touching the extrados at the crown.

We require for practical purposes the ideal arch for a combination of all the above four loads.

(3) *Some properties of the Catenary.* The properties of the catenary are not usually stated in the text-books in a form suited to practical needs.

The equation to the catenary, referred to axis and directrix, is, c being the parameter,

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right), \quad \text{.........(i)},$$

and its arc is

$$s = \frac{c}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right) \quad \text{.........(ii)}.$$

Hence we find at once $y\,\delta x = c\,\delta s$.

If w'' be the weight per cubic foot of material, then

$$w'' y\,\delta x = w'' c\,\delta s \quad \text{.........(iii)},$$

or to load the catenary with a vertical load of w'' per cubic foot up to its directrix is the same thing as adding a load $w''c$ per unit length of arc. This is the proof of (γ) stated on p. 4 above.

Now let w be the weight per cubic foot of the arch proper and τ its depth along the cross-section from intrados to extrados, then the total load per unit length of arc will be

$$w''c + (w - w'')\frac{\tau}{2} + w\,\frac{\tau}{2} = w''(c - \tfrac{1}{2}\tau) + w\tau,$$

and accordingly the horizontal thrust of the arch at any point (or the total thrust at the crown) $= Q_0 = c \times$ weight on the arch per foot run $= c\{w''(c - \tfrac{1}{2}\tau) + w\tau\}$

$$Q_0 = c\{w''(c - \tfrac{1}{2}\tau) + w\tau\} \quad \text{.........(iv)}.$$

Next consider the catenary, a parabola, a circle and an ellipse referred to the ends of the vertical diameters as origin. The equations are

$$y_c = \frac{x^2}{2c} + \frac{x^4}{24c^3} + \frac{x^6}{720c^5} + \text{etc.} \qquad \text{(v)},$$

$$y_p = \frac{x^2}{p} \qquad \text{(vi)},$$

$$y_a = \frac{x^2}{2a} + \frac{x^4}{8a^3} + \frac{x^6}{16a^5} + \text{etc.} \qquad \text{(vii)},$$

$$y_e = \frac{\beta}{2\alpha^2}x^2 + \frac{\beta}{8\alpha^4}x^4 + \frac{\beta}{16\alpha^6}x^6 + \text{etc.} \qquad \text{(viii)},$$

where c is the parameter of the catenary; p, of the parabola; a, the radius of the circle; and β the semi-vertical, α the semi-horizontal axis of the ellipse. Here we have expanded y in powers of x.

Now if we take $p = 2c$, $a = c$ and $\beta/\alpha^2 = 1/c$, $\beta/\alpha^4 = 1/3c^3$, or $\alpha = \sqrt{3}c$ and $\beta = 3c$, we find that the circle fits the catenary worst

$$y_a - y_c = \tfrac{1}{12}\frac{x^4}{c^3} + \text{etc.}$$

Next comes the parabola with

$$y_c - y_p = \tfrac{1}{24}\frac{x^4}{c^3} + \text{etc.},$$

and lastly with the best fit the ellipse

$$y_e - y_c = \tfrac{1}{180}\frac{x^6}{c^5} + \text{etc.},$$

a far higher order of approximation.

Now let us consider the relation of rise to span for these cases. For the catenary, if r = rise and l = span,

$$r = \frac{l^2}{8c} + \frac{l^4}{384c^3} + \text{etc.},$$

hence

$$c = \frac{l^2}{8r} + \tfrac{1}{6}r \text{ approximately} \qquad \text{(ix)}.$$

For the parabola, parameter $2c$,

$$c = \frac{l^2}{8r}, \text{ accurately} \qquad \text{(x)}.$$

For the circle, radius c,

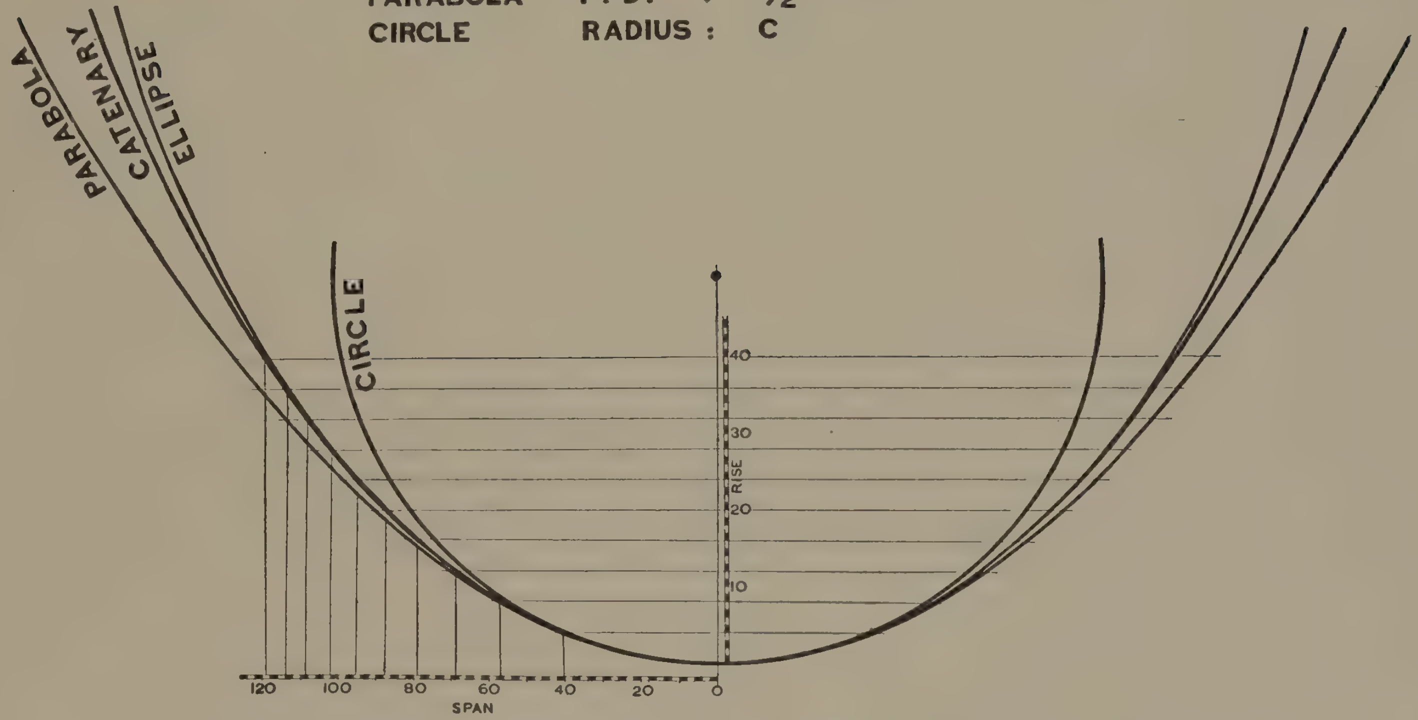
$$c = \frac{l^2}{8r} + \tfrac{1}{2}r, \text{ accurately} \qquad \text{(xi)}.$$

For the ellipse, axes $\sqrt{3}c$ and $3c$,

$$c = \frac{l^2}{8r} + \tfrac{1}{9}r, \text{ accurately} \qquad \text{(xii)}.$$

CATENARY FITTED WITH OTHER CURVES : PARAMETER 20″ = C

ELLIPSE ½ AXES : 3C & √3C
PARABOLA F. D. : C/2
CIRCLE RADIUS : C



Reduced from a lecture-diagram, the span of the catenary being 5 ft. 6 in. The catenary is plotted theoretically, but exactly fitted by a very fine chain suspended from two needles,—all invisible on the photograph,—thus demonstrating the close accordance between theory and experiment.

We can now measure the maximum difference in ordinate between these curves in terms of the ratio of rise to span. We find approximately

$$\frac{\max.\ (y_a - y_c)}{r} = \frac{8}{3}\left(\frac{r}{l}\right)^2; \qquad \frac{\max.\ (y_c - y_p)}{r} = \frac{4}{3}\left(\frac{r}{l}\right)^2;$$

$$\frac{\max.\ (y_e - y_c)}{r} = \frac{128}{45}\left(\frac{r}{l}\right)^4.$$

In an arch of which the rise is $\frac{1}{5}$ of the span, we have

$$\frac{\max.\ (y_e - y_c)}{r} = \frac{1}{220} \text{ nearly.}$$

On the scale of most drawings of arches, this is within the limits of draughtsmanship. We may therefore assert that for all practical purposes the catenary arch, of which the ratio of rise to span is less than $\frac{1}{5}$, may be replaced by an elliptic arch. For most practical cases we may go as far as rise to span $=\frac{1}{4}$. Or we conclude that:

For the fairly flat arches of modern practice designed to carry (i) *a uniform load per foot run of the rib, or* (ii) *a vertical load rising to a horizontal at a height* $l^2/8r + \frac{1}{6}r$ *above the central line at the crown, the elliptic arch is the proper design.*

This proposition relieves us from the trouble of drawing a catenary, or determining a catenary centering. It further places the elliptic arch on a solid foundation of admitted usefulness. We propose in this paper to ascertain the extent of its applicability.

The parabolic arch can hardly be used to replace the catenary arch, until we come to a ratio of rise to span of about $\frac{1}{10}$ at least. It is not till a ratio even less perhaps than this, that we are justified in adopting the ordinary practice of distributing the rib load uniformly along the platform and taking a parabolic arch for the design.

The construction of the elliptic arch requisite for a given rib load $w\tau$ per foot run, and a given load w'' up to the "directrix" is perfectly simple. If r and l be rise and span, we find c from

$$c = \frac{l^2}{8r} + \frac{1}{6}r \ldots\ldots .$$

We then draw an ellipse with vertical semi-axis $3c$ and horizontal semi-axis $\sqrt{3}c$. The resulting horizontal thrust is by (iv):

$$Q_0 = c\,\{w''(c - \tfrac{1}{2}\tau) + w\tau\}.$$

The limitation of this method is that the directrix is not arbitrary*.

* It is worth noting that this resemblance in shape of the elliptic arch and the catenary, enables us very approximately to find the length of an arc of *this* ellipse measured from the vertex to any point, if the subtense of the double arc be not more than $\frac{1}{5}$ of the chord, and very closely indeed, if it be less than $\frac{1}{10}$. For we have by a well-known property of the catenary

$$(\text{Ordinate from directrix})^2 = (\text{arc})^2 + c^2.$$

Or, $$(r+c)^2 = s^2 + c^2, \text{ i.e. } s^2 = r^2 + 2rc.$$

Hence $$s = \sqrt{\tfrac{4}{3}r^2 + \tfrac{1}{4}l^2}.$$

This formula is useful in calculating the total material $2s\tau$ in the arch proper.

In Diagram I we have a photographic reproduction of an actual free hanging uniform chain giving the catenary and illustrating the degree of approximation to this form of the closest circle, parabola and ellipse. It shows within what a considerable range the elliptic arch may be used to replace the appropriate catenary arch.

(4) *Application of the simple Elliptic Arch.* We have already seen that the defect in the elliptic arch so far developed is that for a given rise to span, the height of the vertical load as given by c is determined. But there are a considerable number of cases in which the rise is of relatively small importance, and c may be taken as the quantity which is directly or indirectly given *a priori*. We illustrate this in the following example.

A subway about 20′ broad has to be carried along under a roadway which is 33′ above the bottom of the subway. The subway must have vertical side walls 12′ high and it is needful for the crossing of a main sewer that the crown of the arch should be at least 15′ below the surface. The arch is to consist of 24″ thickness of brick weighing 115 lbs. per cubic foot, and the material up to the roadway may be taken as 100 lbs. per cubic foot. Design an elliptic arch, neglecting backing, to carry the load practically without shear or bending moment.

We have at once by the conditions of the problem

$$c + r = 33 - 12 = 21.$$

But by Equation (ix), p. 6,

$$c = l^2/(8r) + \tfrac{1}{6}r,$$

or

$$21 - r = 50/r + \tfrac{1}{6}r.$$

Thus

$$7r^2 - 126r + 300 = 0,$$

$$r = 15'{\cdot}176, \text{ and } = 2'{\cdot}824.$$

Only the second solution is appropriate. Hence we find

$$c = 18{\cdot}176.$$

This gives:

$$\text{Vertical semi-axis} = 54'{\cdot}528,$$

$$\text{Horizontal semi-axis} = 31'{\cdot}482.$$

An ellipse was determined from these data and the upper part of it gives the proper form for the arch.

Equation (iv) p. 5 gives us for the horizontal thrust

$$\begin{aligned} Q_0 &= 18{\cdot}176\,\{100\,(18{\cdot}176 - 1) + 115 \times 2\} \\ &= 18{\cdot}176 \times 1947{\cdot}6 \\ &= 35{,}400 \text{ lbs.} \end{aligned}$$

Or, since the arch is 2 ft. deep, a compressive stress at the crown of 123 lbs. per sq. inch is reached.

The thrust at any other point distant y from the surface is given by

$$Q = y\{w''(c - \tfrac{1}{2}\tau) + w\tau\}$$
$$= y \times 1947{\cdot}6.$$

The maximum value of y is 12 ft. at the abutments. Hence abutment thrust $= 40{,}880$ lbs.

This corresponds to a compression of 142 lbs. per sq. inch. Now the safe load for good brickwork is just about 140 lbs. per sq. inch.

Hence we may conclude that the arch has been safely designed and the arch proper may be considered as distinctly economic in brickwork. It is figured in Plate I.

(5) *Case of Elliptic Arch with any Span and Rise, designed to carry a load up to any horizontal level*.*

In the previous case we could not consider both c and r as arbitrary. We propose now to allow for this by adding a uniform platform load of $w'\tau'$ per foot run to the loads already considered.

We will first consider the suitable form of a catenary to carry a uniform load $w\tau$ per foot run of the arch and a uniform load $w'\tau'$ per foot run of the horizontal, thus neglecting both the backing and the gravel of the real arch. This load system is of course really more appropriate to a metal rib, but the resulting ideal arch is of some

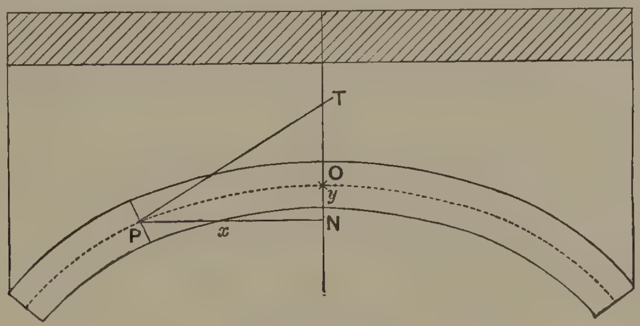


DIAGRAM II.

interest as being intermediate between the parabola and the catenary. Let ϕ be the angle the tangent to the central line at any point P makes with the horizontal, O the point on the central line at the crown, the vertical and horizontal through O the axes of y and x respectively; thus in our figure $ON = y$, $PN = x$, $OP = s$, $\angle TPN = \phi$. If Q be the thrust at P, Q_0 at the crown, we have from considering the equilibrium of the length OP of arch

$$Q \cos\phi = Q_0, \quad Q \sin\phi = sw\tau + xw'\tau'.$$

Hence
$$\tan\phi = \frac{w\tau}{Q_0}s + \frac{w'\tau'}{Q_0}x.$$
$$p = as + bx, \text{ say} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xiii)}.$$

* We are much indebted to Mr E. Cunningham for assistance in the algebra of this section.

Thus
$$\frac{dp}{dx} = a\sqrt{1+p^2} + b,$$

and
$$\frac{dp}{dy} = \frac{a\sqrt{1+p^2}+b}{p}.$$

The latter equation gives us for all values of a and b

$$y = \frac{1}{a}(\sqrt{1+p^2}-1) - \frac{b}{a^2}\log_e\left(\frac{a\sqrt{1+p^2}+b}{a+b}\right) \quad \text{.................(xiv)},$$

and the former

$$x = -\frac{1}{a}\log_e(\sqrt{1+p^2}-p) - \frac{b}{a\sqrt{a^2-b^2}}\sin^{-1}\frac{p\sqrt{a^2-b^2}}{a\sqrt{1+p^2}+b} \quad \text{...........(xv)},$$

if
$$a > b,$$

or,
$$x = -\frac{1}{a}\log_e(\sqrt{1+p^2}-p) - \frac{b}{a\sqrt{b^2-a^2}}\log_e\frac{p\sqrt{b^2-a^2}+a+b\sqrt{1+p^2}}{a\sqrt{1+p^2}+b} \quad \text{...(xvi)},$$

if
$$b > a,$$

or,
$$x = -\frac{1}{a}\log_e(\sqrt{1+p^2}-p) - \frac{1}{a}\frac{p}{1+\sqrt{1+p^2}} \quad \text{.................(xvii)},$$

if
$$b = a.$$

Thus we see that while the form of y does not alter, the form of x varies according as to whether the weight of rib or platform is the greater, i.e. $w\tau >$ or $< w'\tau'$.

An arch plotted from these equations may be spoken of as a *compound catenary arch*.

But the practical engineer is very unlikely to take the trouble to plot the ideal arch from such equations as these except for very special cases. We have accordingly expanded x and y in terms of p starting from the values of $\frac{dp}{dx}$ and $\frac{dp}{dy}$, then integrating and eliminating p. We have if $a/(a+b) = \gamma$,

$$y_i = \frac{ax^2}{2\gamma} + \frac{a^3x^4}{24\gamma^3} + \frac{a^5x^6}{720\gamma^4}(4\gamma-3) + \text{etc.} \quad \text{.................(xviii)}.$$

Now let us compare this equation with that for the ellipse from p. 6, i.e.

$$y_e = \frac{\beta x^2}{2\alpha^2} + \frac{\beta x^4}{8\alpha^4} + \frac{\beta x^6}{16\alpha^6} + \text{etc.},$$

the two first terms will be identical, if we take

$$\beta/\alpha^2 = a/\gamma, \qquad \beta/\alpha^4 = a^3/(3\gamma^3),$$

or
$$\alpha^2 = 3\gamma/a^2, \qquad \beta = 3/a.$$

That is to say, the ellipse of closest fit is the one with semi-horizontal axis and semi-vertical axis given respectively by

$$\alpha = \sqrt{3}/\{a(a+b)\}^{\frac{1}{2}} = \sqrt{3}Q_0/\sqrt{w\tau(w\tau+w'\tau')} \quad \text{.................(xix)},$$
$$\beta = 3/a \qquad = 3Q_0/(w\tau) \quad \text{...............................(xx)}.$$

For the order of approximation, we find

$$y_e - y_i = \frac{\beta x^6}{16\alpha^6}\left(1 - \tfrac{4}{5} + \frac{3}{5\gamma}\right) + \text{etc.} = \frac{\beta x^6}{80\alpha^6}\left(1 + \frac{3}{\gamma}\right), \text{ nearly,}$$

and taking maximum value in terms of rise, we have

$$\frac{\max.\ (y_e - y_i)}{r} = \tfrac{128}{45}\left(\frac{r}{l}\right)^4 \frac{w\tau + \frac{3}{4}w'\tau'}{w\tau + w'\tau'} \quad \text{......................(xxi).}$$

Comparing this with the result on p. 7, we see that the fit is always better than in the case of the simple catenary. We conclude therefore that the properly designed elliptic arch is for most practical purposes the ideal arch to carry a given load per foot run of the rib, and a uniform platform load.

In order to bring the degree of approximation clearly before the reader, Plate II has been drawn giving the approximate ellipse, and the ideal or compound catenary arch plotted from Equations (xiv) and (xvi) for $w\tau/w'\tau' = \frac{1}{4}$.

Plate III compares the elliptic and compound catenary arches, the latter drawn from Equations (xiv) and (xv) for $w\tau/w'\tau' = 2$. We see that for the ratio of rise to span up to ·38 in Plate II and ·25 in Plate III, the elliptic arch answers admirably.

From the Equation (xiv) for y_i we have at once

$$p = (a+b)\,x + \tfrac{1}{6}\,a\,(a+b)^2\,x^3 + \tfrac{1}{120}\,a\,(a+b)^3\,(a-3b)\,x^5,$$

and substituting in the equation $p = as + bx$ we find

$$s = x + \tfrac{1}{6}(a+b)^2\,x^3 + \frac{(a+b)^3\,(a-3b)}{120}\,x^5 \quad \text{..................(xxii),}$$

or, if preferred in terms of the semi-axes

$$s = x + \tfrac{1}{6}\frac{\beta^2}{\alpha^4}\,x^3 + \frac{\beta^2\,(4\alpha^2 - \beta^2)}{40\alpha^8}\,x^5 \quad \text{......................(xxiii).}$$

If x be put equal to the half span, this enables us to find the weight $w\tau \times 2s$ of the arch proper with sufficient accuracy for practical purposes.

We are now in a position to design an elliptic arch of any rise and span to carry a definite platform and rib load.

If r be rise and l span as before,

$$r = \tfrac{1}{8}\frac{\beta}{\alpha^2}\,l^2 + \tfrac{1}{2}\frac{r^2}{\beta}, \text{ as before,}$$

$$= \tfrac{1}{8}(a+b)\,l^2 + \tfrac{1}{6}\,ar^2$$

$$= \tfrac{1}{8}\left(\frac{w\tau}{Q_0} + \frac{w'\tau'}{Q_0}\right)l^2 + \tfrac{1}{6}\frac{w\tau}{Q_0}\,r^2.$$

Thus

$$Q_0 = (w\tau + w'\tau')\,\tfrac{1}{8}\frac{l^2}{r} + w\tau\,\tfrac{1}{6}\,r \quad \text{......................(xxiv).}$$

This gives the horizontal thrust of the arch.

But Q_0 being known α and β follow from Equations (xix) and (xx). The central line of the arch can now be drawn. The total weight of the rib can be calculated from (xxiii), $= 2s\tau w = W$, say, and $W' = lw'\tau'$, the total platform load is known. Hence the maximum thrust Q_m at the abutments is given by

$$Q_m = \sqrt{Q_0^2 + \tfrac{1}{4}(W + W')^2}$$

and the maximum compressive stress can be determined.

As example, let us take the following:

Required an elliptic metal arch, the rib weighing $\frac{1}{2}$ ton per foot run, to carry a 2 tons per foot run platform across a span of 200 ft. with a rise of 25 ft. without bending moment or shear.

Clearly $$Q_0 = (\tfrac{1}{2} + 2)\,\tfrac{1}{8}\,\frac{(200)^2}{25} + \tfrac{1}{2}\,\tfrac{1}{6}\,25 = 502{\cdot}1 \text{ tons.}$$

We then find $$\left.\begin{aligned}\alpha &= 777{\cdot}8 \text{ ft.}\\ \beta &= 3012{\cdot}5 \text{ ft.}\end{aligned}\right\}\beta/\alpha = 3{\cdot}873.$$

Calculating s, the half length of the arc from Equation (xxiii), we find it $= 104{\cdot}02$ ft. Thus the total weight of rib $= 2s \times w\tau = 104{\cdot}02$ tons. The platform load being 400 tons, we have for the total load on the arch 504·02 tons, and the abutment thrusts

$$Q_m = \sqrt{(502{\cdot}1)^2 + (252{\cdot}01)^2} = 561{\cdot}8 \text{ tons.}$$

The maximum stress in the rib can now be ascertained as soon as its cross-section has been determined.

Plate IV gives the form of this elliptic arch and its line of pressure. The latter was obtained in the following manner. A parabola was first drawn corresponding to a vector figure of vertical height 400 tons and polar distance 502·1 tons. Next the arc was broken up into 6 feet elements, each of which weighs 3 tons, and the link polygon drawn for these loads with a polar distance also $= 502{\cdot}1$ tons. This link polygon added to the parabola gives the line of pressure. For accuracy in this case it was found best to plot the ellipse from calculated points. The accordance between the line of pressure and the ellipse measures the degree of confidence we may grant to the theory which replaces the compound catenary of Equations (xiv—xvi) by the appropriate simple elliptic arch. We note that within the errors of draughtsmanship all the points found on the line of pressure lie actually on the ellipse.

(6) *General Case of the Ideal Masonry Arch.* The degree of accordance indicated in the above sections between the elliptic arch and various catenary types would at first sight suggest that an elliptic arch could probably be found so that its central line practically coincided with the line of pressure even in the case of the most general symmetrical system of loading such as we meet with in the ordinary masonry arch.

We must consider this point a little more at length. We have shewn that the elliptic arch works well for a uniform load per foot run of the horizontal and a uniform load per foot run of the rib. We now propose to fill up with backing and gravel the space from the extrados of the arch to the roadway. Practically the

effect of this is to much increase the load at the springings. Now this must mean in general terms that the line of pressure grows much steeper at the abutments, unless we have a very small ratio of rise to span. If we have such a small ratio of rise to span, say $\frac{1}{15}$ or $\frac{1}{20}$, we can still use an elliptic arch with excellent results. But such an arch gives a very large horizontal thrust, and in many cases this will be undesirable, or even impossible. If we have such ratios of rise to span as occur in practice, say $\frac{1}{10}$ or even more, then the line of pressure becomes very steep towards the abutments. The ellipse accordingly will be nearly a semi-ellipse or $\frac{1}{2}l/a$ is not such a small quantity that we can afford to neglect even its sixth power. It results from this that our approximation based upon a neglect of this power fails, and the true line of pressure when drawn deviates from the centre line of the elliptic arch. If we endeavour to make the terms up to x^6 agree in ellipse and true line of pressure by a fit choice of backing our equations will be found to lead us to impossible values—in practice—for the backing. We have tried this on a number of existing arches of usual dimensions, and we think we must conclude that the elliptic arch, while quite as good for a 'flat' masonry arch (allowing for backing and gravel) as for a metal arch, is not suited to masonry arches with a large ratio of rise to span. This result explains fully the instability shewn by the ordinary theory for the elliptic railway arch, used so frequently in practice. We shall endeavour to bring these points out in the following general investigation, where we have determined the constants of the ideal arch in the most general case.

We suppose the following load system, the arch being taken as a strip of $1'$ breadth.

(*a*) $w'\tau'\delta x$ = weight of platform load per run δx.

(*b*) $w''\delta x\,\delta y$ = weight of gravel per area $\delta x\,\delta y$.

Let h be the height from the central line at the crown to the top of the gravel, then if we replace h by $h'' = h + \frac{w'}{w''}\tau'$, we can consider our platform load represented by an additional uniform distribution of gravel and we may throw (*a*) into (*b*) and not trouble further about (*a*) until we come to our final equations.

(*c*) We will suppose the backing reaches up to the curve

$$y' = -\tfrac{1}{2}\tau + e_1x^2 + e_2x^4 + e_3x^6 \quad\text{(xxv)},$$

where τ is the thickness of the arch proper at the crown. By a proper choice of e_1, e_2 and e_3 we can make this a parabola, or very closely a circle or an ellipse. If we make it an ellipse we shall retain one constant at our disposal, besides the usually given height of the backing at the springings. The circular form for the backing has been used in Italian masonry arches and for theoretical purposes is far more easy of analytical treatment than the lines tangential to the extrados often used in English practise.

The weight of the backing shall be $w'''\delta x\,\delta y$ per area $\delta x\,\delta y$.

(*d*) $w\tau ds$ = weight of arch proper per length δs.

We have, if y be the ordinate of the central line of the arch measured from its tangent at the crown, the following system of vertical loads:

$$w''(h''+y)\,\delta x,$$
$$(w'''-w'')(y-y')\,\delta x,$$
$$\left\{(w-w''')\frac{\tau}{2}+w\frac{\tau}{2}\right\}\delta s.$$

Let Q_x be the thrust at x, then as before for the equilibrium of the load on the arc from crown to x, we must have

$$Q_x \cos\phi = Q_0,$$
$$Q_x \sin\phi = (w\tau - \tfrac{1}{2}w'''\tau)\,s + w''\int_0^x (h''+y)\,\delta x + (w'''-w'')\int_0^x (y-y')\,dx.$$

As before take $\tan\phi = p$, and let us write

$$\tau(w-\tfrac{1}{2}w''')/Q_0 = a',$$
$$w''/Q_0 = b', \quad (w'''-w'')/Q_0 = c'.$$

Then we have

$$p = a's + b'h''x + (b'+c')\int_0^x y\,dx - c'\int_0^x y'\,dx \quad \text{.................(xxvi).}$$

Now let us assume that y can be expanded in powers of x, or

$$y = f_1\frac{x^2}{2} + f_2\frac{x^4}{4} + f_3\frac{x^6}{6} + f_4\frac{x^8}{8} + \quad \text{.........................(xxvii),}$$

which gives

$$p = f_1 x + f_2 x^3 + f_3 x^5 + f_4 x^6 + \dots,$$
$$\frac{dp}{dx} = f_1 + 3f_2 x^2 + 5f_3 x^4 + 7f_4 x^6 + \dots.$$

But from Equation (xxvi), we have

$$\frac{dp}{dx} = a'\sqrt{1+p^2} + b'h'' + (b'+c')\,y - c'y',$$
$$= a'\{1 + \tfrac{1}{2}p^2 - \tfrac{1}{8}p^4 + \tfrac{1}{16}p^6\} + b'h'' + (b'+c')\,y - c'y'.$$

Or, keeping terms to x^6,

$$f_1 + 3f_2x^2 + 5f_3x^4 + 7f_4x^6$$
$$= a'\{1 + \tfrac{1}{2}f_1^2x^2 + f_1f_2x^4 + (f_1f_3 + \tfrac{1}{2}f_2^2)\,x^6 - \tfrac{1}{8}f_1^4x^4 - \tfrac{1}{2}f_1^3f_2x^6 + \tfrac{1}{16}f_1^6x^6\}$$
$$+ b'h'' + (b'+c')(\tfrac{1}{2}f_1x^2 + \tfrac{1}{4}f_2x^4 + \tfrac{1}{6}f_3x^6) - c'(-\tfrac{1}{2}\tau + e_1x^2 + e_2x^4 + e_3x^6).$$

Equating the coefficients of powers of x on both sides we have

$$f_1 = a' + b'h'' + \tfrac{1}{2}c'\tau \quad \text{...(xxviii),}$$
$$3f_2 = \tfrac{1}{2}a'f_1^2 + \tfrac{1}{2}(b'+c')\,f_1 - c'e_1 \quad \text{..(xxix),}$$
$$5f_3 = a'(f_1f_2 - \tfrac{1}{8}f_1^4) + \tfrac{1}{4}(b'+c')\,f_2 - c'e_2 \quad \text{.............................(xxx),}$$
$$7f_4 = -a'(\tfrac{1}{2}f_1^3f_2 - \tfrac{1}{16}f_1^6 - f_1f_3 - \tfrac{1}{2}f_2^2) + \tfrac{1}{6}(b'+c')\,f_3 - c'e_3 \quad \text{.........(xxxi).}$$

Now several points flow at once from these results. If r and l be rise and span,

$$r = \tfrac{1}{8} f_1 l^2 + \tfrac{1}{64} f_2 l^4 + \tfrac{1}{384} f_3 l^6 + \tfrac{1}{2048} f_4 l^8 \quad \text{.................(xxxii).}$$

Let us suppose in the first place that f_3 is negligible and that the backing is *a priori* given. Then substituting for f_1 and f_2 in Equation (xxxii) we can determine Q_0 the horizontal thrust. This being found, a', b', and c' are determined and f_1 and f_2 will follow. Now in the case of an ellipse

$$f_1 = \frac{\beta}{\alpha^2}, \quad f_2 = \frac{\beta}{2\alpha^4}, \quad f_3 = \frac{3\beta}{8\alpha^6} \quad \text{....................(xxxii) } bis.$$

It is clear therefore that f_1 and f_2 being known α and β will be determined, i.e. the semi-axes of the ellipse. Now the ratio of the term neglected to the term retained is

$$\tfrac{1}{384} f_3 l^6 / \tfrac{1}{64} f_2 l^4 = \tfrac{1}{2} \left(\frac{\frac{1}{2} l}{\alpha} \right)^2.$$

Unless therefore α is large as compared to the half span, the ellipse as calculated from the first two terms only of Equation (xxvii) cannot possibly give a good approximation to the true line of pressure of the arch.

If we consider the backing at our disposal and go as far as f_3, we have two constants e_1 and e_2 at our choice; we can use one of these to give the backing an arbitrary height at the springings, and the other to make f_3 zero or very small. If this be done while f_1 and f_2 will give possibly a close approximation to the form of the curve, that curve will not be an ellipse since f_3 will fail to equal $3\beta/8\alpha^6$ as required.

It seems very desirable to illustrate this point fully. Neglecting f_3 we have

$$r = \tfrac{1}{8} f_1 l^2 + \tfrac{1}{64} f_2 l^4 \quad \text{.............................(xxxiii).}$$

Let us write $\chi_2 = l/r$ = ratio of span to rise, and $\chi_3 = r/\tau$ = ratio of rise to thickness of arch.

Further if ρ be the radius of curvature of the backing at the crown, $2e_1$ is the curvature $= 1/\rho$. We will take $\chi_1 = \rho/\tau$, and therefore $e_1 = 1/(2\chi_1\tau)$.

Now from Equation (xxviii)

$$\begin{aligned} f_1 &= \{\tau(w - \tfrac{1}{2}w''') + w''h'' + \tfrac{1}{2}(w''' - w'')\tau\}/Q_0 \\ &= \left\{\tau(w - \tfrac{1}{2}w'') + \left(h + \frac{w'}{w''}\tau'\right) w''\right\} \Big/ Q_0 \\ &= \tau\left(w + \frac{\tau'}{\tau} w' + \frac{h - \frac{1}{2}\tau}{\tau} w''\right) \Big/ Q_0 \\ &= \tau(w + m'w' + m''w'')/Q_0 \quad \text{..................................(xxxiv).} \end{aligned}$$

Now take $$\overline{w} = w + m'w' + m''w'' \quad \text{.............................(xxxv),}$$

and, $$Q_0 = \overline{w}\tau^2 q_0 \quad \text{...(xxxvi).}$$

Hence $$f_1 = \frac{1}{q_0 \tau} \quad \text{...(xxxvii).}$$

Here $\overline{w}$ is a known and calculable quantity for a given bridge for:

(i) w = weight per cubic foot of masonry of arch proper.

(ii) $m'w'$ = contribution due to uniform horizontal load $= \tau' w'/\tau$ = weight per foot square of horizontal load divided by thickness of arch proper.

(iii) $m''w''$ = weight per cubic foot of gravel $\times m''$, or the ratio of the depth from extrados at crown to the top of the gravel to the thickness of the arch proper $(m'' = (h - \frac{1}{2}\tau)/\tau)$.

But $\overline{w}$ being found, it is clear that the determination of the horizontal thrust will depend upon a knowledge of q_0, which is obviously a pure numerical quantity, $\overline{w}\tau^2$ being a force.

From Equation (xxix) we have

$$f_2 = \tfrac{1}{6}\left(\frac{a'}{q_0^2\tau^2} + \frac{(b'+c')}{q_0\tau} - \frac{c'}{2\chi_1\tau}\right).$$

If we write

$$n_1 = w'''/\overline{w}, \quad n_2 = (w - \tfrac{1}{2}w''')/\overline{w}, \quad n_3 = (w''' - w'')/\overline{w} \quad \text{.........(xxxviii)},$$

we find after some reductions

$$f_2 = \frac{1}{6q_0^2\tau^3}\left(n_1 + \frac{n_2}{q_0} - \frac{n_3 q_0}{\chi_1}\right) \quad \text{........................(xxxix)}.$$

Substituting in the value for r, Equation (xxxiii), we have

$$1 = \tfrac{1}{8}\chi_2^2\chi_3\frac{1}{q_0} + \tfrac{1}{384}\frac{\chi_2^4\chi_3^3}{q_0^2}\left(n_1 + \frac{n_2}{q_0} - \frac{n_3 q_0}{\chi_1}\right),$$

or

$$q_0^3 - q_0^2\left(\tfrac{1}{8}\chi_2^2\chi_3 - \frac{n_3}{384}\frac{\chi_2^4\chi_3^3}{\chi_1}\right) - \frac{n_1}{384}\chi_2^4\chi_3^3 q_0 - \frac{n_2}{384}\chi_2^4\chi_3^3 = 0 \quad \text{...........(xl)},$$

this is a cubic to find q_0. If q_0 be found from this equation, f_1 and f_2 and Q_0 can be found, and from Equations (xxxii *bis*) the semi-axes α and β of the ellipse.

For a ratio of rise to span such as we get in many metal arches this process gives very good results. It is not, however, satisfactory in the case of such rise to span as we find for current design in brick and stone arches. There is no difficulty in locating the root of the cubic. To a first approximation $q_0 = \frac{1}{8}\chi_2^2\chi_3$ and to a second

$$q_0 = \tfrac{1}{8}\chi_2^2\chi_3 + \tfrac{1}{8}\chi_3\left(\tfrac{1}{8}n_1\chi_2^2\chi_3 + n_2 - \tfrac{1}{8}n_3\frac{\chi_2^4\chi_3^2}{\chi_1}\right).$$

Taking $q_0 = \overline{q}_0 + \epsilon$, where $\overline{q}_0$ is the above value and ϵ a small quantity, we easily find ϵ by Newton's process.

Illustration. We take the following arch, a modified form of a Chester bridge.

Rise (height of central line at crown, above centre line at springings) $= 10' = r$. Span $= 100' = l$. Thickness of arch proper $= 3{\cdot}5' = \tau$. Height of road level above central line at crown $= 3{\cdot}5' = h$. Backing circular, and cutting vertical through centre line at springings at a height of $5'$. Thus $\rho = \dfrac{50^2 + (6{\cdot}75)^2}{2 \times 6{\cdot}75} = 188{\cdot}56'$. Live load $= 120$ lbs.

per sq. foot of roadway. Gravel weighs 110 lbs. per cubic foot. Backing weighs 140 and arch ring 160 lbs. per cubic foot.

$$w'm' = \tau' w'/\tau = 120/3{\cdot}5 = 34{\cdot}3 \text{ lbs. per sq. ft.},$$

$$m'' = (h - \tfrac{1}{2}\tau)/\tau = (3{\cdot}5 - 1{\cdot}75)/3{\cdot}5 = {\cdot}5,$$

$$\overline{w} = w + m'w' + m''w'' = 160 + 34{\cdot}3 + {\cdot}5 \times 110 = 249{\cdot}3 \text{ lbs. per sq. ft.}$$

Again
$$n_1 = w'''/\overline{w} = {\cdot}56157,$$

$$n_2 = (w - \tfrac{1}{2}w''')/\overline{w} = {\cdot}36101,$$

$$n_3 = (w''' - w'')/\overline{w} = {\cdot}12034,$$

$$\chi_1 = \rho/\tau = 53{\cdot}886, \quad \chi_2 = l/r = 10, \quad \chi_3 = r/\tau = 2{\cdot}857.$$

The cubic is easily found to be

$$q_0^3 - 34{\cdot}3563 q_0^2 - 341{\cdot}0385 q_0 - 219{\cdot}2395 = 0.$$

The root is at once localised between 42 and 43, and starting with $q_0 = 42{\cdot}5 + \epsilon$ we find

$$q_0 = 42{\cdot}5018.$$

Then we have
$$f_2 = \frac{{\cdot}07919}{q_0^2 \tau^3} = \frac{\beta}{2\alpha^4}, \quad f_1 = \frac{1}{q_0 \tau} = \frac{\beta}{\alpha^2},$$

thus
$$\alpha^2 = \frac{q_0 \tau^2}{{\cdot}15838} = 3287{\cdot}3282,$$

leading to
$$\alpha = 57{\cdot}335 \text{ and } \beta = 22{\cdot}099.$$

It will be at once seen that α is only 7 feet more than the half span and the ratio of the term neglected in the value of r to the term retained $= \frac{1}{2}\left(\frac{50}{57{\cdot}335}\right)^2 = {\cdot}38$. The approximation is thus not close enough. In fact the values of f_1 and f_2 being substituted we have

$$y = 8{\cdot}403 \left(\frac{x}{50}\right)^2 + 1{\cdot}597 \left(\frac{x}{50}\right)^4 \quad \text{(xli)}.$$

This gives for $x = 50$, the true rise $y = 10$. But the actual ellipse with $\alpha = 57{\cdot}335$ and $\beta = 22{\cdot}099$ gives for a rise of 10, the span of about 96. In other words two terms of the expansion of y in terms of x, do not sufficiently closely represent the ellipse. Such an ellipse as the above fails to represent the facts, i.e. has no approach to the ideal arch. We may next inquire whether an equation like (xli) above, which is not an ellipse, represents closely enough the ideal arch. To determine this we have to plot Equation (xli) and then draw the true line of pressure, assuming the horizontal thrust is $Q_0 = q_0 \tau^2 \overline{w} = 129{,}797$ lbs. This is done in Plate V. It will be seen that the pseudo-ellipse (whole line) given by the second approximation does not differ visibly, except towards the springings, from the central line of the arch (broken line). Below we have indicated the wide deviation of the ellipse from the pseudo-ellipse (xlv). On

the right is drawn the critical line of pressure; it clearly differs very widely from the approximate form determined by the pseudo-ellipse, giving too small a horizontal thrust and much too large stresses.

We could make our ellipse and our ideal arch curve agree up to terms in x^6 by selecting the e_2 of our backing so that f_1, f_2 and f_3 satisfied the elliptic relation $f_2^2 = 3f_1 f_3$, but trial has shewn that this often gives a very unsuitable contour for the backing, which it seems desirable to leave in some simple form, as the circular. The above illustration brings out our point that the elliptic arch is not a sufficiently good approximation (for the current values of rise and span) to the ideal arch.

(7) *Higher Approximation to the Form of the Ideal Arch.*

We shall now take account of the f_3 terms in the rise and span equation, i.e.

$$r = \tfrac{1}{8} f_1 l^2 + \tfrac{1}{64} f_2 l^4 + \tfrac{1}{384} f_3 l^6.$$

The first point is to determine f_3. Now from Equation (xxx):

$$f_3 = \tfrac{1}{5}\{a'(f_1 f_2 - \tfrac{1}{8} f_1^4) + \tfrac{1}{4}(b' + c') - c' e_2\}.$$

Now for the backing

$$y' = -\tfrac{1}{2}\tau + e_1 x^2 + e_2 x^4 + \dots$$

$$= -\tfrac{1}{2}\tau + \tau \left\{\tfrac{1}{2}\phi_1 \left(\frac{x}{\tau}\right)^2 + \tfrac{1}{8}\phi_2 \left(\frac{x}{\tau}\right)^4 + \dots\right\} \quad \text{.....(xlii)},$$

where

$$\phi_1 = 2e_1\tau, \quad \phi_2 = 8e_2\tau^3 \quad \text{.....(xliii)}.$$

Thus

$$\phi_1 = \tau/\rho = 1/\chi_1 \quad \text{.....(xliv)},$$

of our previous section,

$$\phi_2 = (\tau/\rho)^3 = \phi_1^3 = 1/\chi_1^3 \quad \text{.....(xlv)},$$

if the backing be as usual circular in contour.

Remembering that

$$f_1 = \frac{1}{q_0 \tau}, \quad f_2 = \frac{1}{6 q_0^2 \tau^3}\left\{n_1 + \frac{n_2}{q_0} - n_3 q_0 \phi_1\right\} \quad \text{.....(xlvi)},$$

and the values of a', $b' + c'$, and c' we find

$$f_3 = \tfrac{1}{5}\left\{\frac{n_2}{\tau q_0}\left[\frac{1}{6 q_0^3 \tau^4}\left(n_1 + \frac{n_2}{q_0} - n_3 q_0 \phi_1\right) - \tfrac{1}{8}\frac{1}{q_0^4 \tau^4}\right] + \tfrac{1}{4}\frac{n_1 f_2}{\tau^2 q_0} - \frac{n_3}{\tau^2 q_0}\tfrac{1}{8}\frac{\phi_2}{\tau^3}\right\},$$

or,

$$f_3 = \frac{1}{q_0^5 \tau^5}\{\tfrac{1}{30} n_2^2 - \tfrac{1}{40} n_2 + \tfrac{1}{24} n_1 n_2 q_0 + (\tfrac{1}{120} n_1^2 - \tfrac{1}{30} n_2 n_3 \phi_1) q_0^2 - \tfrac{1}{120} n_1 n_3 \phi_1 q_0^3 - \tfrac{1}{40} n_3 \phi_2 q_0^4\}$$
$$\text{......(xlvii)}.$$

We must now substitute in Equation (xxxii) for f_1, f_2, f_3 noting that as before $\chi_2 = l/r$, $\chi_3 = r/\tau$, but writing for brevity

$$\lambda_1 = 64/(\chi_2^6 \chi_3^5) = 64\frac{r}{l}\left(\frac{\tau}{l}\right)^5 \quad \text{.....(xlviii)},$$

$$\lambda_2 = 2/(\chi_2 \chi_3) = 2\tau/l \quad \text{.....(xlix)}.$$

We find

$$\lambda_1 = \tfrac{1}{2}\frac{\lambda_2^4}{q_0} + \tfrac{1}{24}\frac{\lambda_2^2}{q_0^2}\left\{n_1 + \frac{n_2}{q_0} - n_3 q_0 \phi_1\right\}$$

$$+ \tfrac{1}{6}\frac{1}{q_0^5}\left\{\tfrac{1}{30}n_2^2 - \tfrac{1}{40}n_2 + \tfrac{1}{24}n_1 n_2 q_0 + \left(\tfrac{1}{120}n_1^2 - \tfrac{1}{30}n_2 n_3 \phi_1\right) q_0^2 - \tfrac{1}{120}n_1 n_3 \phi_1 q_0^3 - \tfrac{1}{40}n_3 \phi_2 q_0^4\right\}.$$

Multiplying up and rearranging we have to find q_0 the equation of the fifth order

$$\begin{aligned} \lambda_1 q_0^5 - q_0^4 &\left\{\tfrac{1}{2}\lambda_2^4 - \tfrac{1}{24}n_3\phi_1\lambda_2^2 - \tfrac{1}{240}n_3\phi_2\right\} \\ - q_0^3 &\left\{\tfrac{1}{24}n_1\lambda_2^2 - \tfrac{1}{720}n_1 n_3 \phi_1\right\} \\ - q_0^2 &\left\{\tfrac{1}{24}n_2\lambda_2^2 + \tfrac{1}{720}n_1^2 - \tfrac{1}{180}n_2 n_3\phi_1\right\} \\ - q_0 &\left\{\tfrac{1}{144}n_1 n_2\right\} + \tfrac{1}{240}n_2 - \tfrac{1}{180}n_2^2 = 0 \quad \text{.......................(l).} \end{aligned}$$

In solving this equation, the coefficients will all be found to be very small for such arches as occur in practice; it is therefore desirable to calculate them in the form $\frac{1}{10^6}$ (numerical quantity). A first rough approximation to the desired root is

$$q_0 = \tfrac{1}{2}\lambda_2^4/\lambda_1 = \tfrac{1}{8}\chi_2^2\chi_3 = \tfrac{1}{8}\frac{l}{r\tau} \quad \text{..........................(li),}$$

of our previous investigation. But since the terms are, except the constant term, usually all negative we require a value somewhat, perhaps 25 °/。, in excess of this. Giving integer values to q_0 of this order we rapidly locate the required root, and find its value by Newton's method. q_0 being known, f_1, f_2 and f_3 can be calculated and so a high approximation to the perfect arch form can be obtained. A few lines of analysis then demonstrate whether this form can or cannot be effectively given by an ellipse. If the ellipse be not close enough, it is always possible to represent the proper curve as the combination of an ellipse and a parabola by a method indicated below. But we have found in actual draughtsmanship that the calculation of the ordinates of the perfect arch from the Equation (xxvii) is really the shortest and most effective method of design.

Illustration. We have shewn that the elliptic arch is not a good approximation, even when the rise is only $\frac{1}{10}$ of the span. We shall see that the present method gives excellent results when the proportions are much closer to those of current practice.

Let us take the following constants for our arch. Rise = 15′, span = 100′. Thickness of arch ring 3·5′. Backing contour circular and rising 10′ above central line at springings. Hence $\rho = 188{\cdot}56$. Weights of live-load, gravel, backing and arch ring as in previous illustration. Height of road level above central line at crown = 3·5′.

Thus as before we have

$$\overline{w} = 249{\cdot}3 \text{ lbs. per sq. ft.,} \quad n_1 = {\cdot}56157, \quad n_2 = {\cdot}36101, \quad n_3 = {\cdot}12034.$$

$\chi_1 = 53{\cdot}886$, but $\chi_2 = 6{\cdot}6667$ and $\chi_3 = 4{\cdot}2857$.

We easily find

$$\lambda_2 = \cdot 070{,}000, \quad \lambda_1 = \cdot 000{,}000{,}50421,$$

$$\phi_1 = 3\cdot 5/188\cdot 56 = \cdot 018{,}5617,$$

$$\phi_2 = \cdot 000{,}006{,}395{,}19.$$

Calculating the coefficients of the quintic (l) we have after multiplying by 10^6,

$$\cdot 50421 q_0^5 - 11\cdot 54572 q_0^4 - 112\cdot 91157 q_0^3 - 507\cdot 228 q_0^2 - 1407\cdot 864 q_0 + 780\cdot 168 = 0.$$

The first approximation to q_0 is $11\cdot 54572/0\cdot 50421 = 24$ say. 25 °/。 increase is roughly 30. We tried $q_0 = 30$, 31 and 32 and found the sign changed between 31 and 32. Using Newton's method the root required is

$$q_0 = 31\cdot 1999,$$

whence for the thrust of the arch

$$Q_0 = q_0 r^2 \overline{w} = 95{,}282 \text{ lbs.}$$

The values of f_1, f_2, f_3 were now determined and we have for the equation to the required centre line of the arch ring

$$y = 11\cdot 44693 \left(\frac{x}{\frac{1}{2}l}\right)^2 + 3\cdot 14129 \left(\frac{x}{\frac{1}{2}l}\right)^4 + \cdot 41178 \left(\frac{x}{\frac{1}{2}l}\right)^6 \quad \text{...............(lii),}$$

where $\frac{1}{2}l = 50'$ = the half-span. This is a convenient form for calculation.

It is clear that the third term contributes at a maximum about 3 °/。 to the value of y—an amount quite visible on a good drawing—and in the actual arch providing a distance of about 5″ at the springings. The actual middle third of the arch ring being 14″, the distance from centre line to extrados middle third is 7″, and this defect of 5″ would thus be very visible on the drawings, if the third term were neglected.

Let us next consider how this approximation to the ideal arch can be represented by an ellipse. We must compare it with

$$y = \frac{\beta}{2a^2} x^2 + \frac{\beta}{8a^4} x^4 + \frac{3\beta}{48a^6} x^6.$$

Using the first two coefficients we have

$$\frac{\beta}{2a^2} = \frac{11\cdot 44693}{(50)^2}, \quad \frac{\beta}{8a^4} = \frac{3\cdot 14129}{(50)^4},$$

leading to $a = 47\cdot 723$ and $\beta = 20\cdot 856$.

But using these values the ellipse gives

$$y = 11\cdot 44693 \left(\frac{x}{\frac{1}{2}l}\right)^2 + 3\cdot 14129 \left(\frac{x}{\frac{1}{2}l}\right)^4 + 1\cdot 88429 \left(\frac{x}{\frac{1}{2}l}\right)^6,$$

or a rise of 16·47′ as against the actual rise of 15′. Clearly the ellipse is an unworkable approximation to the perfect arch, and ought under no circumstances to be used for the central line of a masonry arch of anything like the dimensions under discussion. It is sufficient to notice that the major axis of the ellipse is less than the whole span to realise how impossible the ellipse as an approximation to such an arch must be.

The actual ellipse as calculated from the first and second terms never reaches the springings!

If we take the ellipse that fits the second and third term of our arch equation we have

$$\alpha = 97{\cdot}650, \quad \beta = 365{\cdot}601,$$

giving in the expanded form

$$y_e = 47{\cdot}9253 \left(\frac{x}{\frac{1}{2}l}\right)^2 + 3{\cdot}14129 \left(\frac{x}{\frac{1}{2}l}\right)^4 + {\cdot}41178 \left(\frac{x}{\frac{1}{2}l}\right)^6,$$

and shewing enough convergence to stop at $(x/\frac{1}{2}l)^6$ for practical purposes. Comparing the first term with that of the equation to the ideal arch we see that

$$y_i = y_e - 36{\cdot}4784 \left(\frac{x}{\frac{1}{2}l}\right)^2.$$

But

$$y_p = 36{\cdot}4784 \left(\frac{x}{\frac{1}{2}l}\right)^2$$

is a parabola, and we see that the ideal arch has for its ordinates the differences of an ellipse and a parabola. This is one method of describing it; but the large values of the axes of the ellipse will generally preclude the use of any convenient graphical method of construction, and we have found it better and quite easy to plot the ordinates of the desired centre line as in Equation (lii).

This pseudo-elliptic arch to a third approximation is illustrated by the continuous line on the left of Plate VI. It leaves the centre line (broken line) slightly towards the quarter span, the deviation being somewhat exaggerated by the engraver to shew the separate curves (let the reader test with the dividers between the *F* and *P* of the "of pressure," printed above, and he will find that the central line is not truly central!). On the whole it must be said that the arch has been well designed so that centre line and true line of pressure closely agree. The critical line of pressure shews that the old method would have reached a horizontal thrust about 6 °/。 to 7 °/。 too small, and stresses 100 °/。 too large.

We venture to term the arches the centre lines of which are of the form

$$y = \tfrac{1}{2} f_1 x^2 + \tfrac{1}{4} f_2 x^4 + \tfrac{1}{6} f_3 x^6,$$

where we keep or neglect the x^6 term, *Pseudo-elliptic Arches.* From the consideration of a number of numerical illustrations we have convinced ourselves that the ideal arch only approaches the circular or elliptic form when either the ratio of rise to span is much less than is usual in masonry design, or the ballast rises to a far greater height above the crown of the arch than often occurs.

(8) *General Conclusions.*

(i) While the parabola is the ideal arch for a uniform horizontal load and a catenary for a uniform load along the rib, yet for most practical purposes an ellipse can be used instead of the catenary. Further for a fairly 'flat' arch to carry both

uniform load on the rib and uniform load along the horizontal, an ellipse is practically the ideal arch.

(ii) The elliptic arch can further be used for a vertical load rising from the arch to a given horizontal.

(iii) The previous conclusions suggest that the elliptic arch may approximate to the ideal arch, when the load consists of (a) the arch ring, (b) the backing, (c) the ballast and (d) a uniform horizontal load, such as are usual in the case of masonry arches. The elliptic arch is found to closely approximate—with right choice of the axes—to the ideal arch even in this case, *if the ratio of rise to span be small.* Such a ratio, however, involving large horizontal thrust and great compressive stresses, is not suited to the brick arches of customary practice.

(iv) The elliptic arch—still less the circular arch—does not seem to approximate to the ideal arch in the case of the usual type of brick arch.

(v) A close approximation however to the arch whose line of pressure coincides with its centre line can be found with no very great labour of calculation. Such an arch we term a pseudo-elliptic arch.

(vi) The pseudo-elliptic arch shows us: (a) that the customary process of drawing a "critical line of pressure" may be very misleading. Applied to an arch the centre line of which is practically its line of pressure, it gives too small a horizontal thrust, (b) it gives compressive stresses which may be 100 °/ₒ more than the actual stresses.

(vii) We believe that the best method in any case is to draw the pseudo-elliptic arch for the given span, rise and load system, and that even should it be needful to somewhat modify the arch ring from the form obtained by the pseudo-elliptic centre line, yet that line will be a far closer approach to the true line of pressure than the so-called "critical line." In fact the theory of the "critical line of pressure" has never had a really sound scientific basis, and it would be better to attempt in each case to design the arch according to the pseudo-elliptic curve which will make the arch practically its own line of pressure, and give a far closer value to the true thrust and stresses in the arch, than it is possible to obtain by the construction of the critical line of pressure, which at best does not provide the actual stresses in the arch, but those which it is supposed would arise if the arch were on the point of collapsing*.

* The authors will be glad if any designer who finds a difficulty in determining the form of the pseudo-elliptic arch for given data will communicate with them at University College, London.

DESCRIPTION OF PLATES.

Plate I. This plate shews the construction of an ellipse which is practically identical with the *Simple Catenary* destined to carry without shear or bending moment, (i) a uniform load per foot run of the arc, and (ii) a uniform load per square foot of area up to the roadway which coincides with the directrix of the catenary. In this drawing no account is taken of any separate backing. The ellipse has been constructed graphically, and the construction is given on the drawing. The thrusts are plotted vertically from the centre line of arch to the curve marked as curve of thrust.

Plate II. This plate illustrates the closeness of agreement between the *Compound Catenary* and the appropriate ellipse for the case of a metal rib, when the dead load per foot run of the horizontal is four times the rib weight per foot run.

Plate III. This plate illustrates the closeness of agreement between the *Compound Catenary* and the appropriate ellipse when the weight of the arch much exceeds, i.e. is double, the dead load. The case is of course unusual, but it shews that the ellipse is practically the true ideal arch even for a rise $\frac{1}{4}$ of the span, in such cases.

Plate IV. In this case an elliptic arch has been designed, rise $\frac{1}{8}$ of span, allowing for rib load and platform load; and then the true line of pressure constructed *a posteriori*, by adding the parabola for the platform load to the link polygon for the rib-load. All the points thus obtained for the true line of pressure are found to lie within the errors of draughtsmanship on the elliptic arch as designed.

Plate V. A Pseudo-Elliptic Arch has been designed of the form $y = \frac{1}{2}f_1x^2 + \frac{1}{4}f_2x^4$, with rise $\frac{1}{10}$ of span. This involves the solution of a cubic equation, when weights of arch, backing, gravel, and platform load are all taken into account. The true line of pressure then is constructed and is found to differ only moderately and near the springings from the designed centre line. The ellipse corresponding to this pseudo-ellipse is drawn and found to diverge excessively from the true line of pressure. On the right-hand side of the diagram is shewn the construction of the "critical line of pressure" for this arch, which in the particular case touches practically the intrados middle third at the springings. This gives a critical horizontal thrust about 8 °/。 less than the true value, but owing to the eccentricity of this critical line of pressure at the crown and springings, the stress values at these sections are between 80 °/。 and 100 °/。 larger than the true values.

Plate VI. A Pseudo-Elliptic Arch has been designed of the form $y = \frac{1}{2}f_1x^2 + \frac{1}{4}f_2x^4 + \frac{1}{2}f_3x^6$ (i.e. to a third approximation) with a rise more than $\frac{1}{7}$ of span. This involves the solution of an equation of the fifth order, when weights of arch, backing, gravel and platform load are all taken into account. The true line of pressure has been constructed *a posteriori* and found to differ hardly at all from the designed centre line—indeed the engraver* to shew the difference between the two has rather exaggerated the deviation shewn on the original drawing. On the right is drawn the so-called critical line of pressure, again touching the intrados middle third almost at the springing. It gives as the arch of Plate V, a considerably reduced horizontal thrust, but stresses increased 80 °/。 to 100 °/。 beyond the true values.

* He omitted centre line and placed it in on the engraving after completion.

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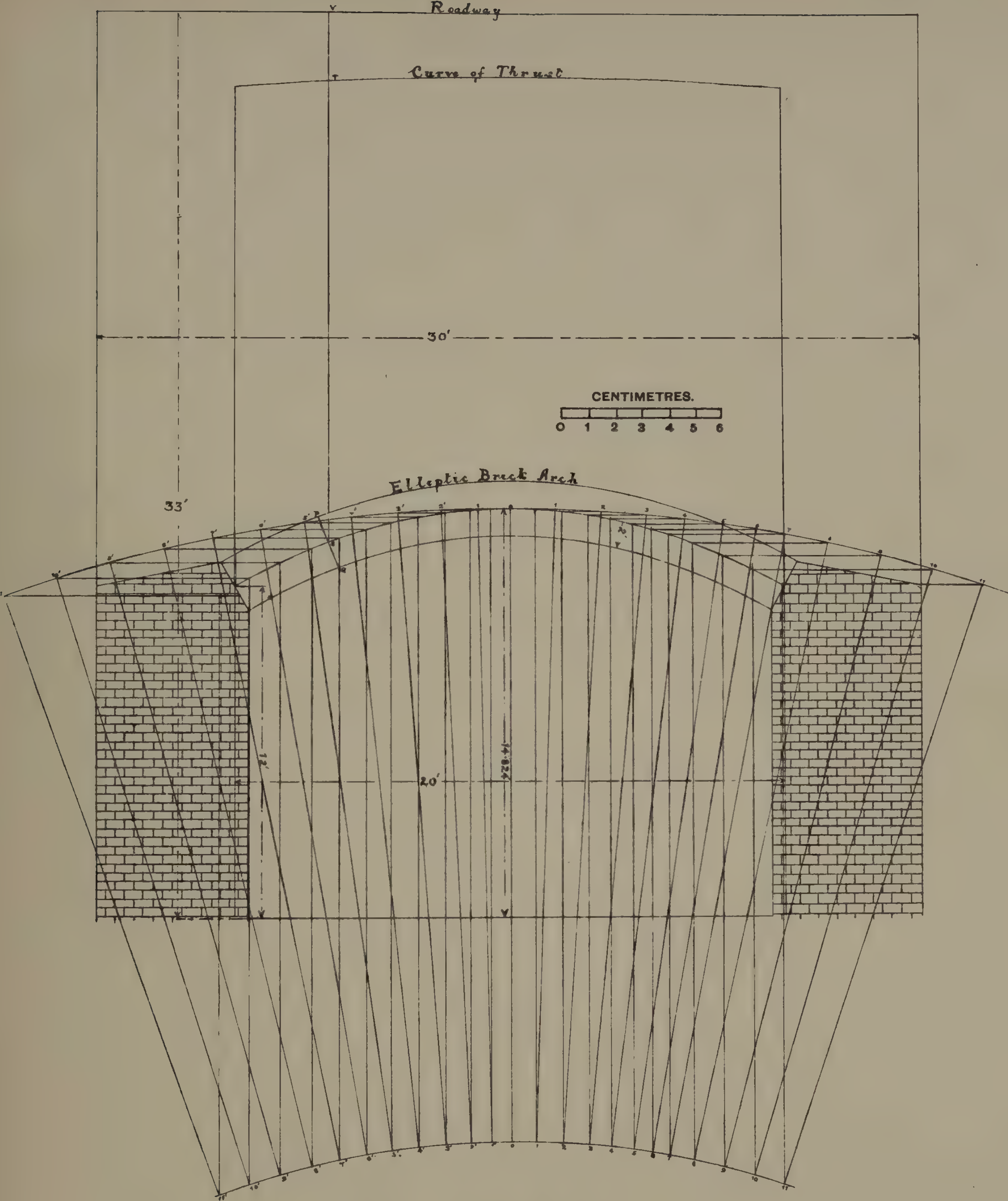


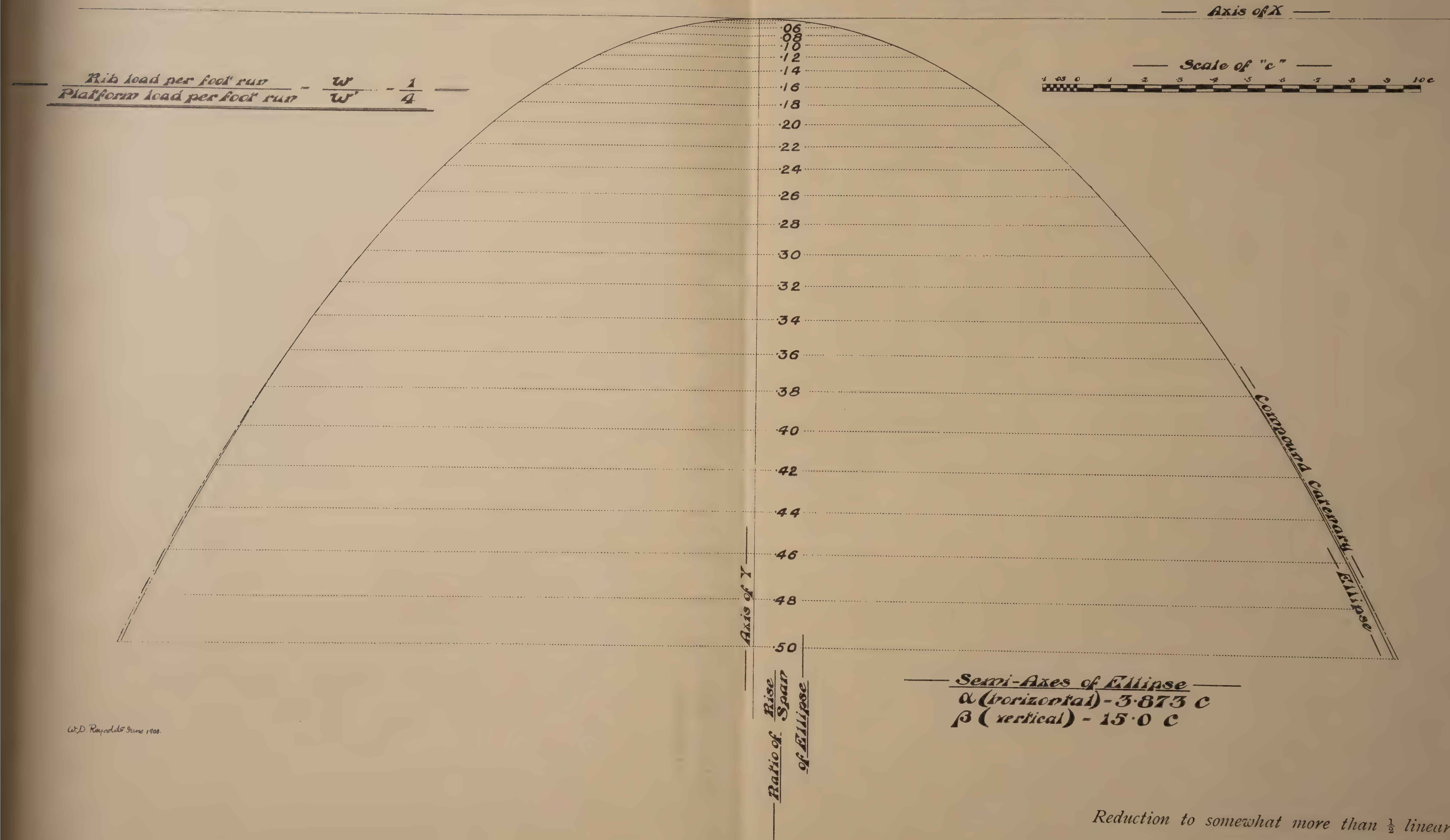
—20' SUBWAY—

— Scales — — Space – 1 cm – 1' –
— Thrust – 1 cm – 1 ton –
— Horizontal Thrust – 15·804 tons —
— Maximum Thrust — 18·28 tons i.e. 142·2 lbs/inch² —
— Shear & Bg. Mt — Nil —

W. D. Reynolds June '05.

Reduction to slightly less than ½ linear.

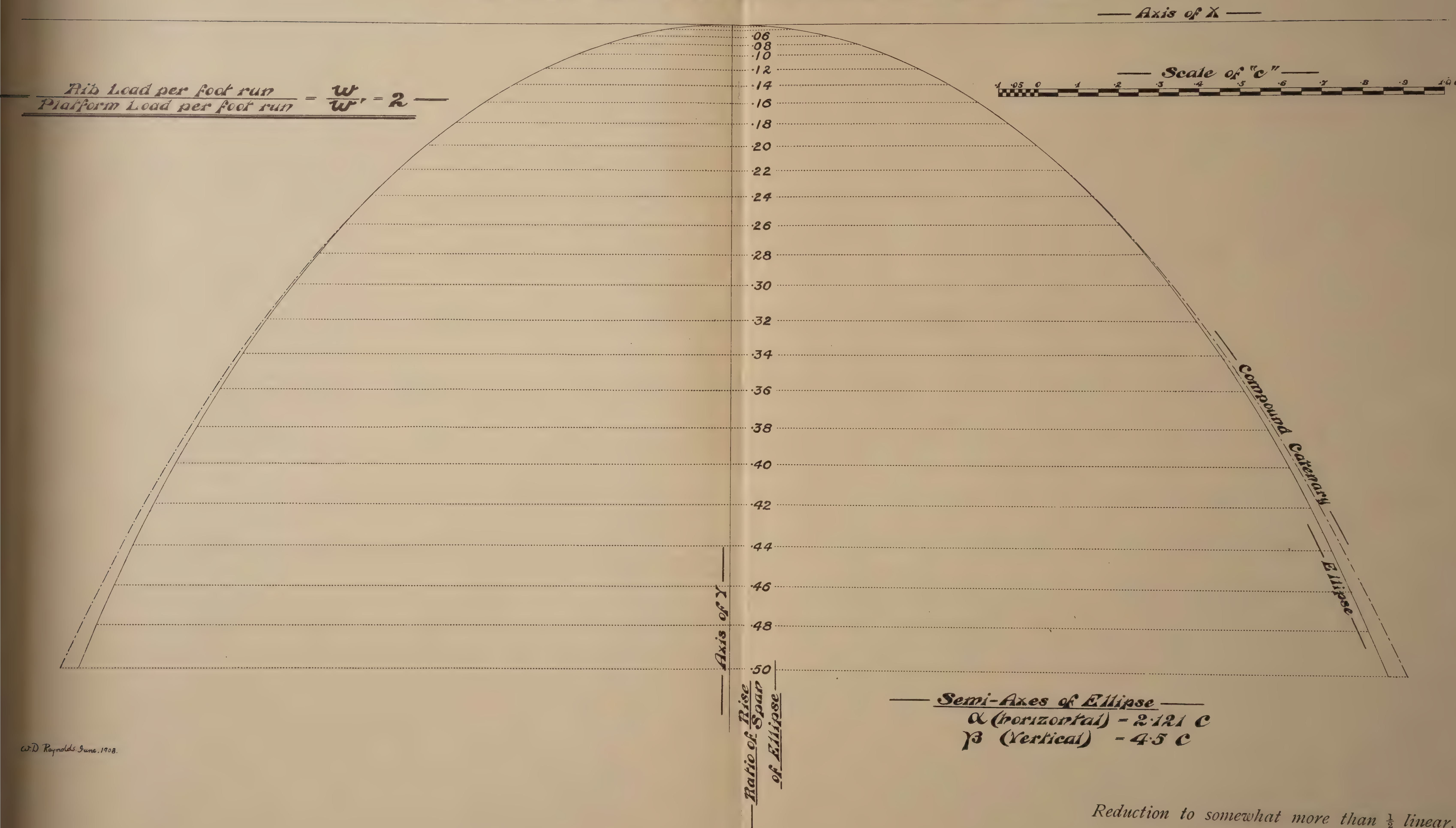
COMPOUND CATENARY ARCH WITH APPROXIMATE ELLIPSE



Reduction to somewhat more than ½ linear

PLATE III.

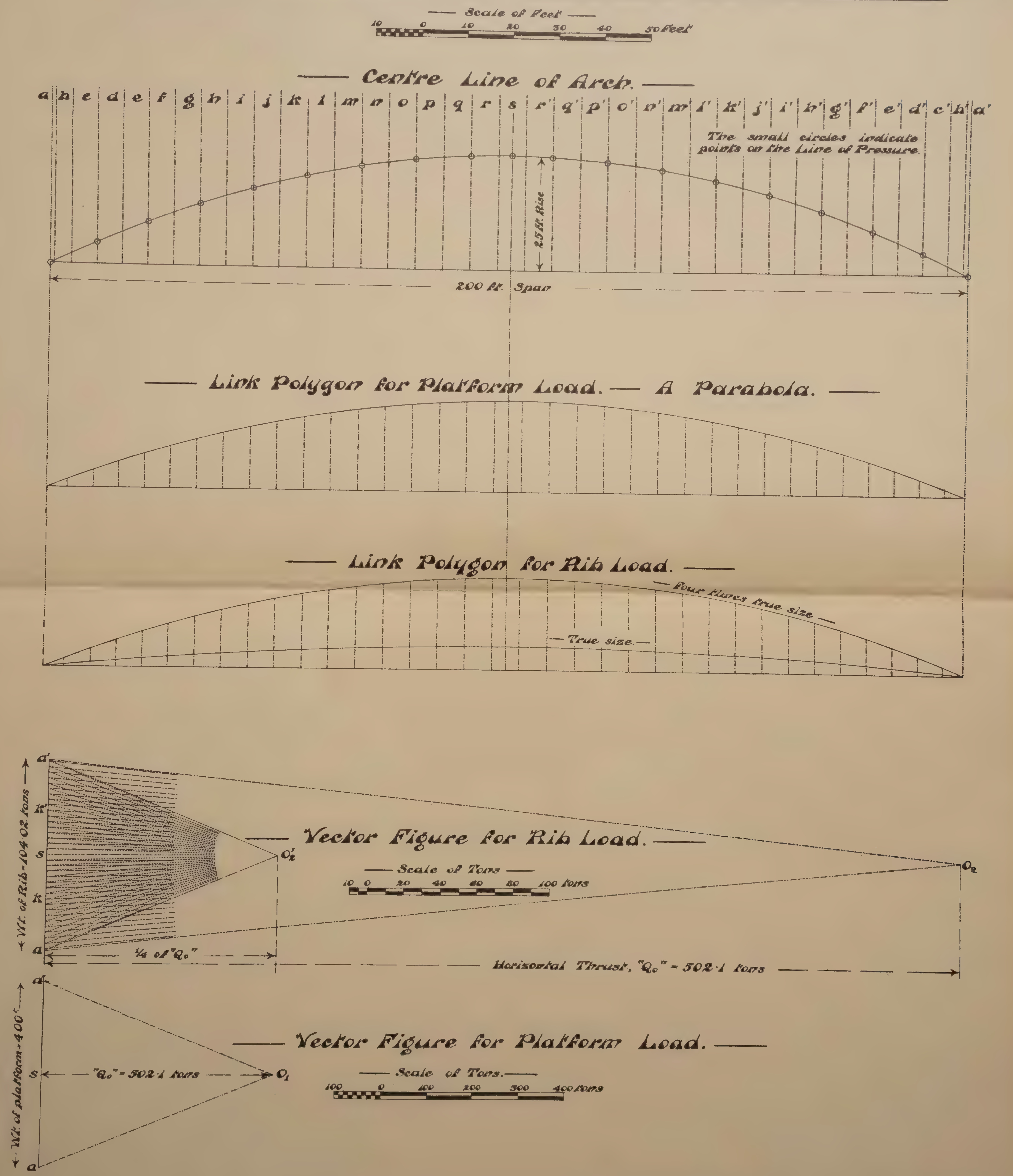
COMPOUND CATENARY ARCH WITH APPROXIMATE ELLIPSE



Reduction to somewhat more than ½ linear.

ELLIPTIC METAL ARCH

having Line of Pressure practically identical with Centre Line.



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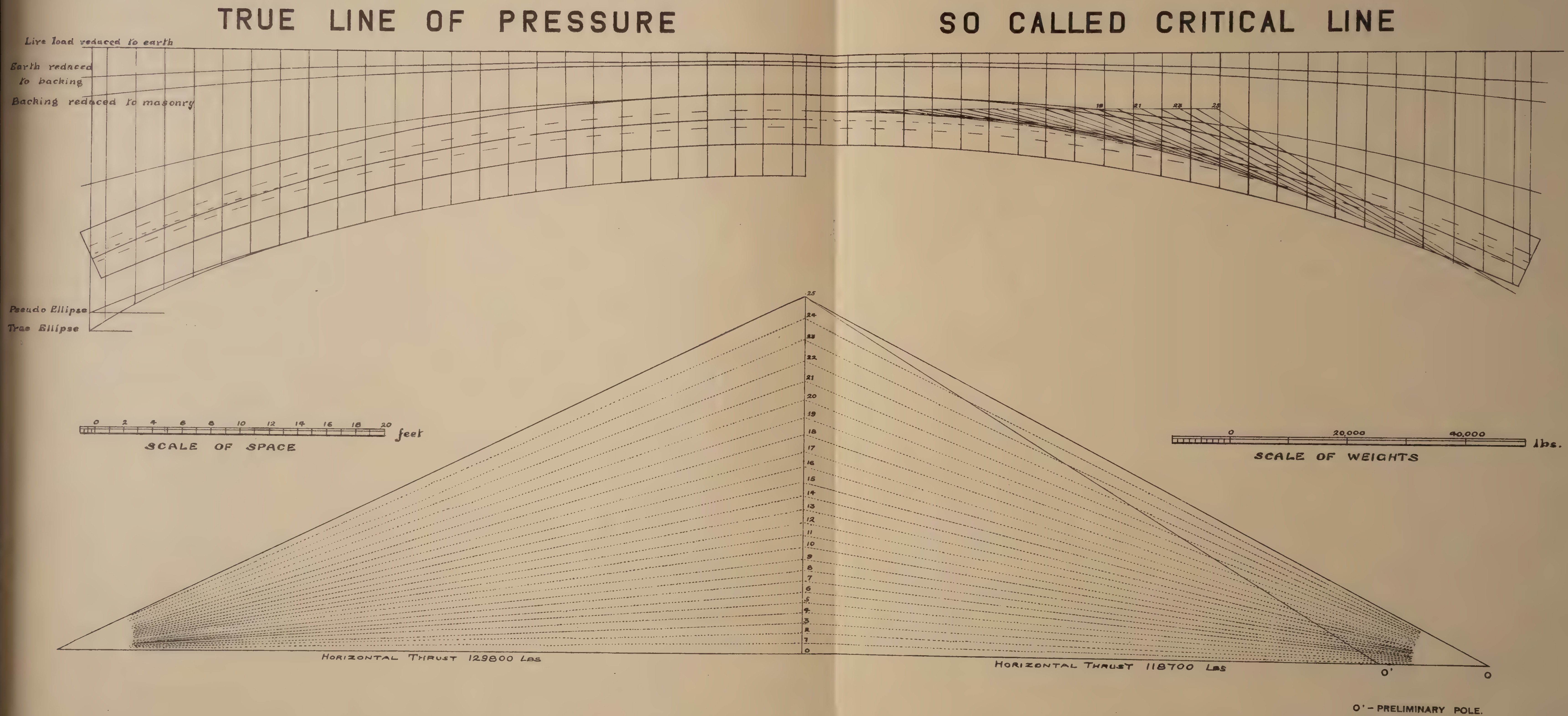
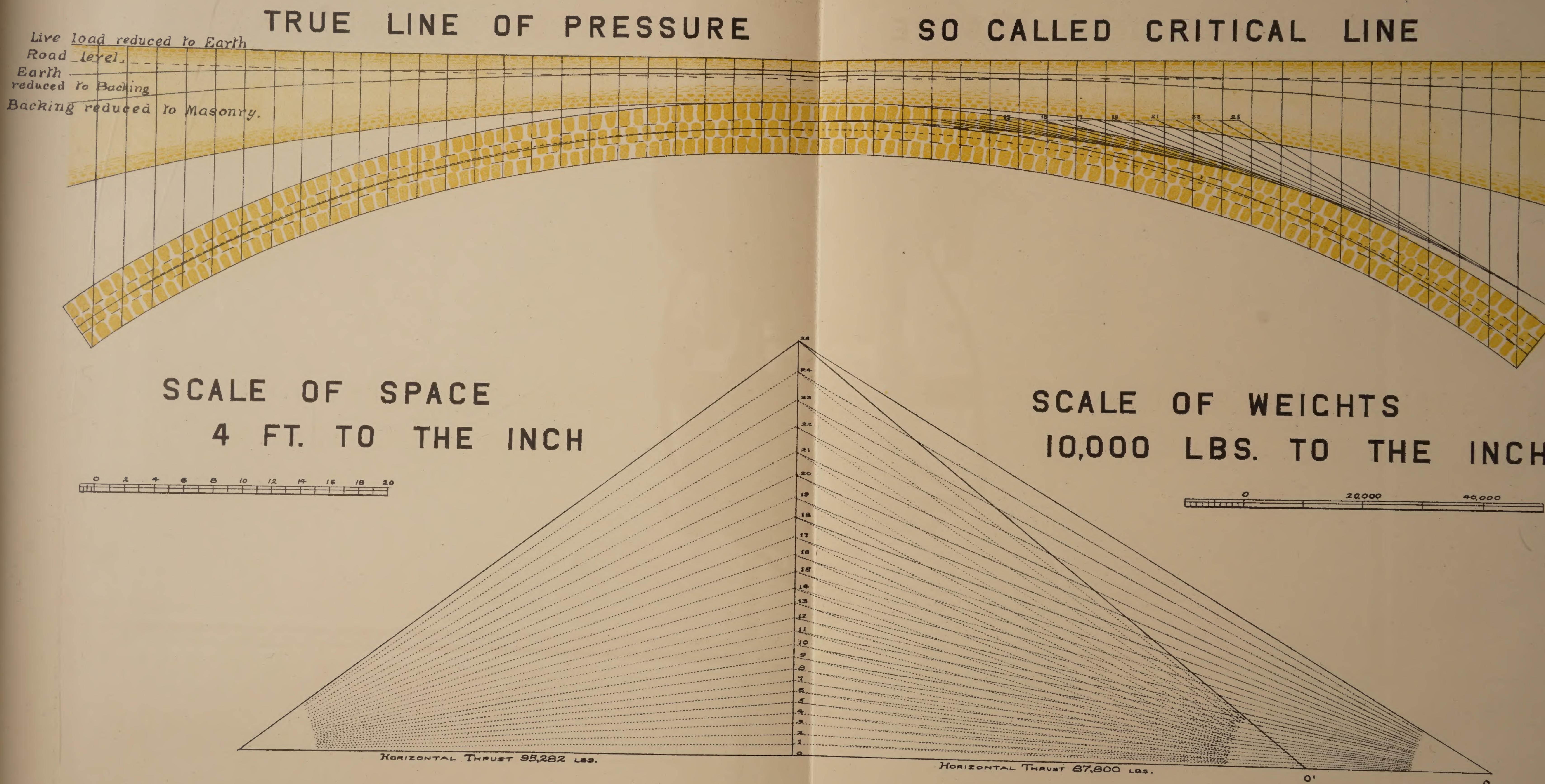


PSEUDO ELLIPTIC ARCH 2ND APPROX.

DESIGNED TO BRING LINE OF PRESSURE ALONG CENTRE LINE

W.F. Stanton
June 1909

Reduction to somewhat more than ½ linear.



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